$Interpolation$

$From\ test,\ a\ set\ of\ data\ are\ obtained;$

$(x_o, y_o), (x_1, y_1), (x_2, y_2), \ldots\ldots\ldots\ldots\ldots\ldots (x_n, y_n)$

$(with\ n+1\ points),\ construct\ a\ polynomial$

$of\ degree\ (n)\ to\ pass\ through\ the\ above\ points:$

$$f(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots\dots + a_n x^n$$

$The\ coefficients\ (a_o, a_1, a_2, \dots\dots\dots\dots a_n)\ are$

$obtained\ from\ the\ data\ above;$

$$(x_o, y_o), (x_1, y_1), (x_2, y_2), \dots\dots\dots\dots (x_n, y_n)$$

$$(x_o, y_o);$$

$$y_o = a_o + a_1 x_o + a_2 x_o^2 + a_3 x_o^3 + \ldots\ldots\ldots\ldots + a_n x_o^n \quad \ldots\ldots(1)$$

$$(x_1, y_1);$$

$$y_1 = a_o + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + \ldots\ldots\ldots\ldots + a_n x_1^n \quad \ldots\ldots(2)$$

$$(x_2, y_2);$$

$$y_2 = a_o + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 + \ldots\ldots\ldots\ldots + a_n x_2^n \quad \ldots\ldots(3)$$

$$\vdots$$

$$\vdots$$

$$(x_n, y_n);$$

$$y_n = a_o + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \ldots\ldots\ldots\ldots + a_n x_n^n \quad \ldots\ldots(n+1)$$

$$1-\ No.\ of\ \text{variables}\ (n+1)\ (a_o,\ a_1,\ a_2,\ldots\ldots\ldots a_n)$$

$$2-\ No.\ of\ equations(n+1)$$

$$3-\ Solve\ by\ (Gauss-Elimination),\ as\ example$$

Example: The following data are obtained from test:

x_i : 0 1 3 4

y_i : 1 2 10 17

Required: *i) Construct a polynomial equation passing through the points*

ii) $f(x)$ *at* $x = 2$

Solution:

$$f(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3$$

$(0,1);$

$$1 = a_o + a_1(0) + a_2(0)^2 + a_3(0)^3 ....(1)$$

$(1,2);$

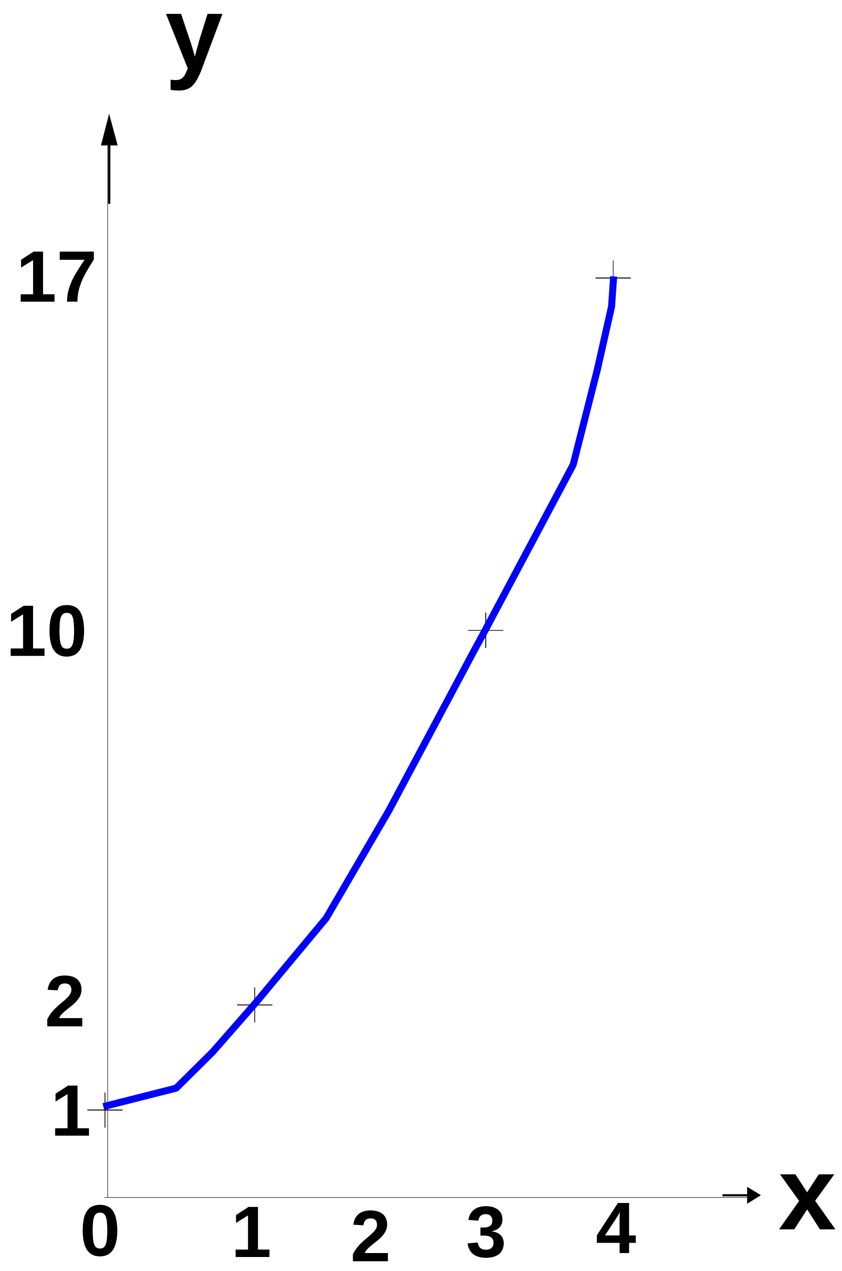
$$2 = a_o + a_1(1) + a_2(1)^2 + a_3(1)^3 ....(2)$$

$(3,10);$

$$10 = a_o + a_1(3) + a_2(3)^2 + a_3(3)^3 ....(3)$$

$(4,17);$

$$17 = a_o + a_1(4) + a_2(4)^2 + a_3(4)^3 ....(4)$$


y
17
10
2
1
x
0
1
2
3
4

$$a_o = 1,$$

$$a_1 + a_2 + a_3 = 1$$

$$3a_1 + 9a_2 + 27a_3 = 9$$

$$4a_1 + 16a_2 + 64a_3 = 16$$

Solve the above equations; get;

$$a_1 = 0, \quad a_2 = 1 \quad and \quad a_3 = 0$$

$$a_1 = 0, \quad a_2 = 1 \quad and \quad a_3 = 0$$

$$\therefore y = f(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3$$

$$= 1 + x^2$$

$$At \quad x = 2 \Rightarrow f(x) = 1 + (2)^2 = 5$$

Differences of Interpolating polynomials:

1 – Newton forward Interpolting:

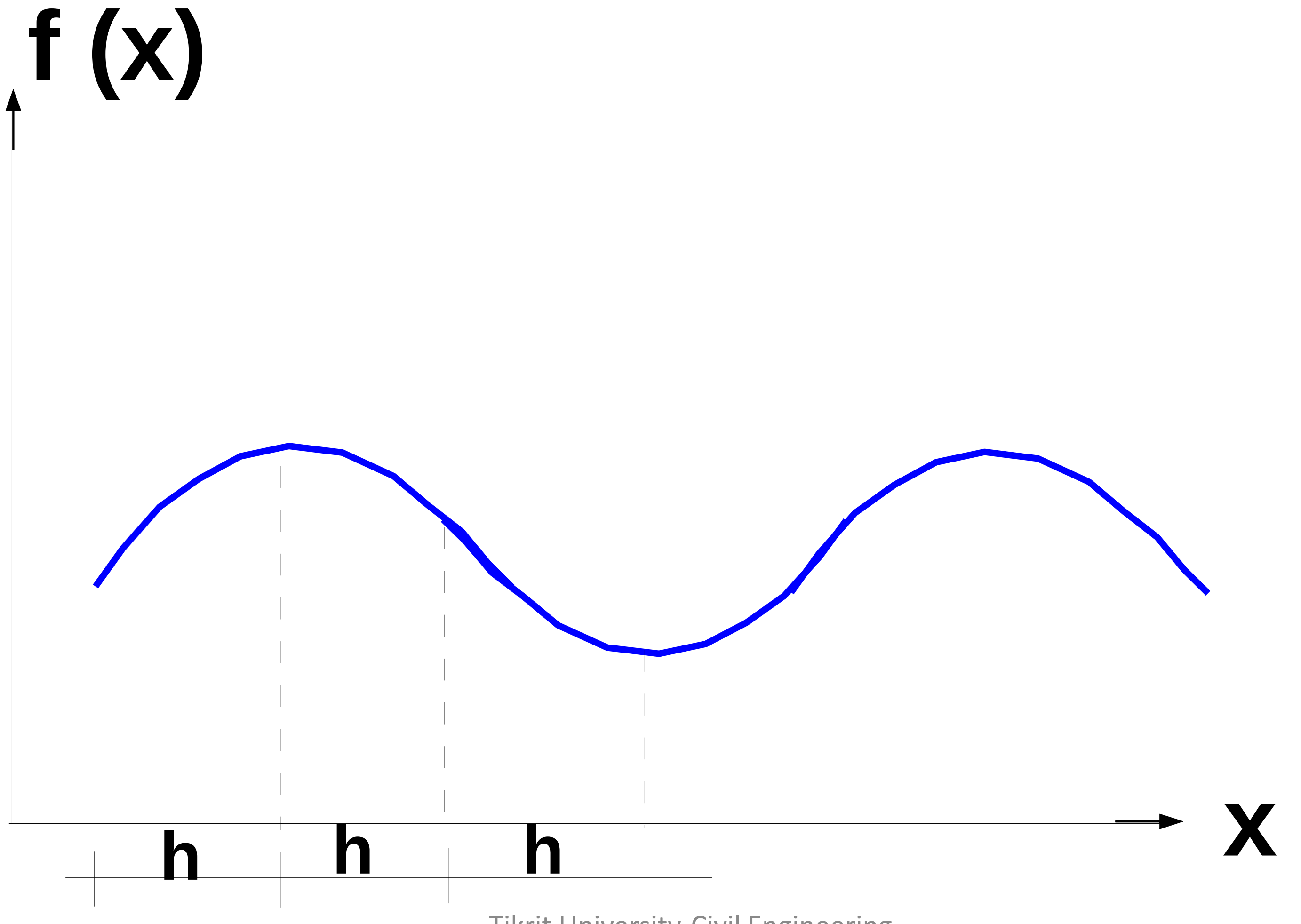
Suppose that a table relating a dependent variable $f(x)$ to an independent variable x is given as:

$$x_i : \quad x_o \quad x_1 \quad x_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad x_n$$

$$f(x_i) : f(x_o) \quad f(x_1) \quad f(x_2) \ldots\ldots\ldots\ldots \quad f(x_n)$$

$$\Delta f x_i = f_{i+1} - f_i$$

$$h = x_{i+1} - x_i \quad (equal\ \text{int}\,ervals)$$



x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	$f(x_0)$				
		Δf_0			
x_1	$f(x_1)$		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	$f(x_2)$		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	$f(x_3)$		$\Delta^2 f_2$		
		Δf_3			
x_4	$f(x_4)$				

$$f(x+h)=Ef(x_i)$$

$$E = Shift \quad operator$$

$$f(x+2h)=E^2 f(x_i)$$

$$f(x+\alpha h)=E^{\alpha} f(x_i)$$

$$E = 1+\Delta$$

$$\alpha = \frac{x-x_i}{h}$$

$$f(x+\alpha h)=E^{\alpha} f(x_i)=(1+\Delta)^{\alpha} f(x_i)$$

By bionomial formula:

$$f(x+\alpha h)=[1+\Delta\alpha+\Delta^2\frac{\alpha(\alpha-1)}{2!}+$$

$$\Delta^3\frac{\alpha(\alpha-1)(\alpha-2)}{3!}+\ldots\ldots]\,f(x_i)$$

$2-Newton\ \ Backward\ \ Interpolting:$

$$E = 1 - \Delta$$

$$f(x+\alpha h) = [1 + \Delta\alpha + \Delta^2 \frac{\alpha(\alpha+1)}{2!} + \Delta^3 \frac{\alpha(\alpha+1)(\alpha+2)}{3!} + \ldots\ldots]\ f(x_i)$$

$Example: Given\ the\ following\ data, approximate$
$the\ functional\ value\ at\ (x = 1.5)$

x_i :	0	1	2	3	4
y_i :	−1	0	3	8	15

Solution

x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	-1			
		1		
1	0		2	
		3		0
2	3		2	
		5		0
3	8		2	
		7		
4	15			

$$1-Forward:\quad x=1.5 \qquad h=1$$

$$\alpha=\frac{x-x_i}{h}\ ;\quad x_i=0$$

$$\alpha=\frac{1.5-0}{1}=1.5$$

$$f(x+\alpha h)=f(1.5)=[1+\alpha\Delta+\Delta^2\frac{\alpha(\alpha-1)}{2!}]f(x_i)$$

$$f(x)=-1,\quad \Delta f(x)=1 \quad and \quad \Delta^2 f(x)=2$$

$$f(1.5)=[-1+1.5*1+2*\frac{1.5(1.5-1)}{2}]=0.5+0.75=1.25$$

$$1-Backward: \quad x=1.5 \qquad h=1 \qquad x_i=3$$

$$\alpha=\frac{x-x_i}{h} \;;$$

$$\alpha=\frac{1.5-3}{1}=-1.5$$

$$At \quad x=3$$

$$f(x)=8, \quad \Delta f(x)=5 \quad and \quad \Delta^2 f(x)=2$$

$$f(1.5)=[1+\alpha$$

$$f(1.5)=[1+\Delta\alpha+\Delta^2\frac{\alpha(\alpha+1)}{2!}]f(x_i)$$

$$f(1.5)=[8+5*(-1.5)+2*\frac{-1.5(-1.5+1)}{2}]$$

$$=8+7.5+0.75=1.25$$

$3 - Lagrange\ Interpolation$

$Consider\ (x_o, y_o), (x_1, y_1), (x_2, y_2) \ldots\ldots\ldots (x_n, y_n)$

$Lagrange\ Interpolation\ \ Polynomial\ is:$

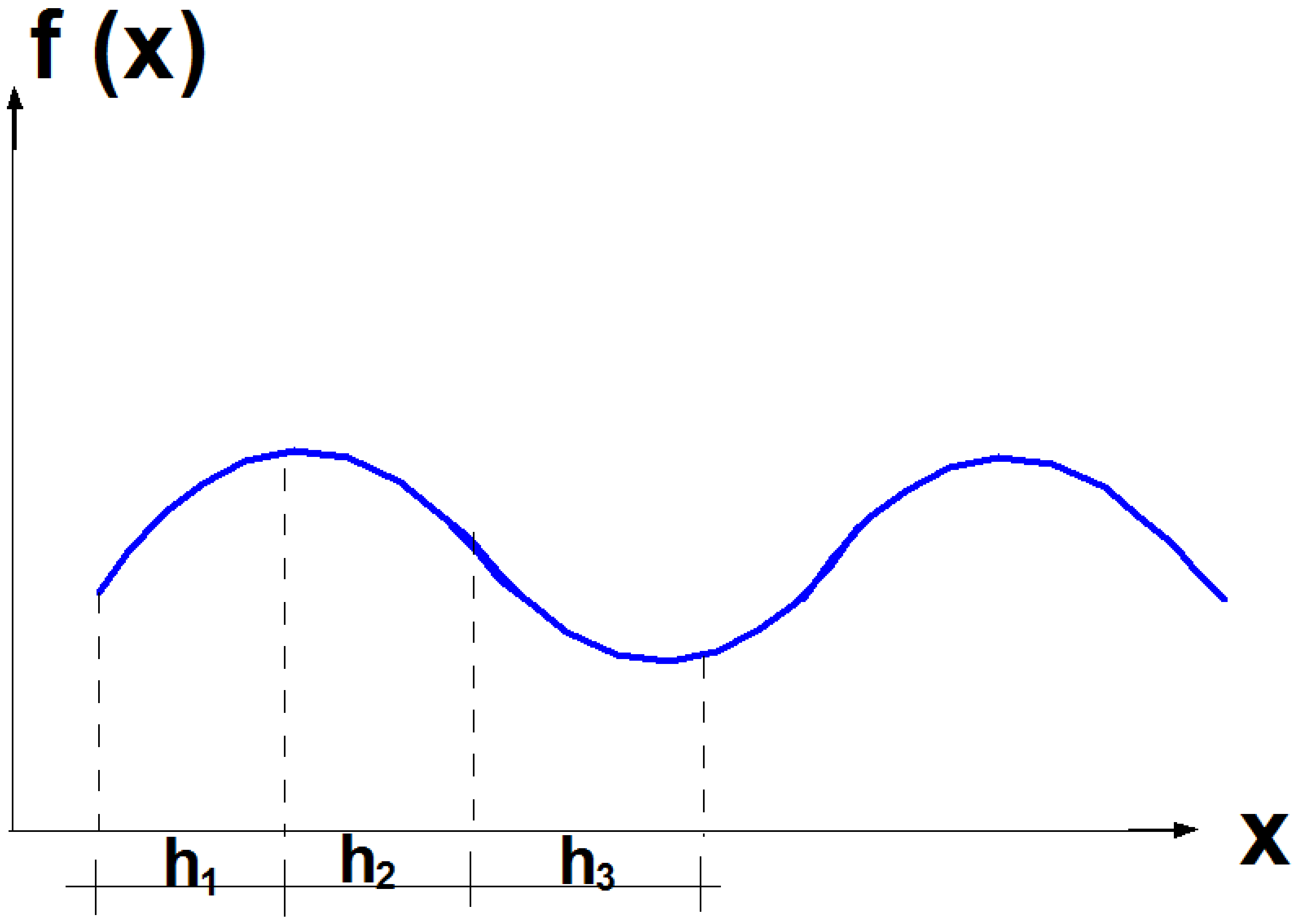
$$f(x) = Ln(x) = \sum_{k=0}^{k=n} \frac{\ell_k(x)}{\ell_k(x_k)} y_k$$

$$\ell_o = (x - x_1)(x - x_2)(x - x_3) \ldots\ldots$$

$$\ell_1 = (x - x_o)(x - x_2)(x - x_3) \ldots\ldots$$

$$\vdots$$

$$\ell_3 = (x - x_o)(x - x_1)(x - x_2)(x - x_4) \ldots\ldots$$


f (x)
x
h_1
h_2
h_3

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)....}{(x_o - x_1)(x_o - x_2)(x_o - x_3).......} * y_o +$$

$$\frac{(x - x_o)(x - x_2)(x - x_3)....}{(x_1 - x_o)(x_1 - x_2)(x_1 - x_3).......} * y_1 +$$

$$\frac{(x - x_0)(x - x_1)(x - x_3)....}{(x_2 - x_o)(x_2 - x_1)(x_2 - x_3).......} * y_2 + ...$$

$$(x = x_o) \Rightarrow f(x) = y_o \quad ; (x = x_1) \Rightarrow f(x) = y_1$$

$Or:$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)....(x-x_n)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)....(x_o-x_n)} * y_o +$$

$$\frac{(x-x_o)(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)....(x_1-x_n)} * y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)....(x-x_n)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)....(x_2-x_n)} * y_2 + ...$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)....(x-x_{n-1})}{(x_n-x_o)(x_n-x_1)(x_n-x_3)....(x_n-x_{n-1})} * y_n$$

$Example: Given$

$$x_i: \quad 0 \quad 2 \quad 3 \quad 5$$

$$y_i: \quad 1 \quad 7 \quad 13 \quad 31$$

$\text{Re}quired: \; f(x) \; at \; x = 4$

$$y = f(4) = \frac{(4-2)(4-3)(4-5)}{(0-2)(0-3)(0-5)} * 1 + \frac{(4-0)(4-3)(4-5)}{(2-0)(2-3)(2-5)} * 7$$

$$+ \frac{(4-0)(4-2)(4-5)}{(3-0)(3-2)(3-5)} * 13 + \frac{(4-0)(4-2)(4-3)}{(5-0)(5-2)(5-3)} * 31$$

$$= \frac{2*1*(-1)}{-2*(-3)(-5)} * 1 + \frac{4*1*(-1)}{2*(-1)(-3)} * 7 + \frac{4*2*(-1)}{3*1*(-2)} * 13 + \frac{4*2*1}{5*3*2} * 31$$

$$= \frac{-2}{-30} * 1 + \frac{-4}{6} * 7 + \frac{-8}{-6} * 13 + \frac{8}{30} * 31 = 21$$

$Example: Given$

x_i :	1	2	3	5
y_i :	0	3	8	14

$\text{Re}\,quired:\ f(x)\ at\ x = 4$

$$y = f(4) = \frac{(4-2)(4-3)(4-5)}{(1-2)(1-3)(1-5)} * 0 + \frac{(4-1)(4-3)(4-5)}{(2-1)(2-3)(2-5)} * 3$$

$$+ \frac{(4-1)(4-2)(4-5)}{(3-1)(3-2)(3-5)} * 8 + \frac{(4-1)(4-2)(4-3)}{(5-1)(5-2)(5-3)} * 24$$

$$= 0 + \frac{3*1*(-1)}{1*(-1)(-3)} * 3 + \frac{3*2*(-1)}{2*1*(-2)} * 8 + \frac{3*2*1}{4*3*2} * 24$$

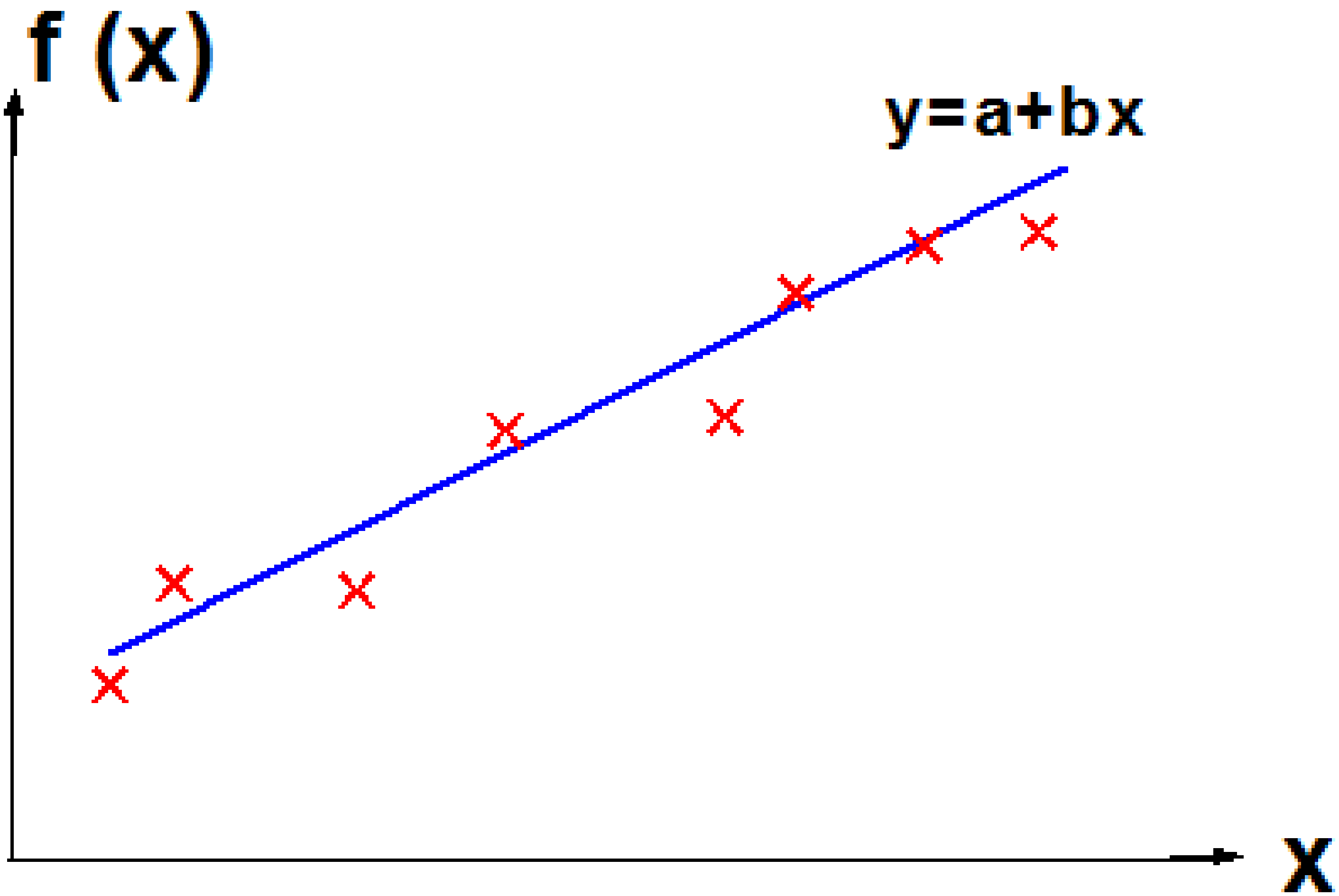
$$= \frac{-3}{3} * 3 + \frac{-6}{-4} * 8 + \frac{6}{24} * 24 = 15$$

Curve Fitting

The method of Least squares

Consider a sample of $(x_1, y_1),(x_2, y_2),(x_3, y_3)...(x_n, y_n)$ of size (n). It is rwquired to draw a best line passing the points above, such that the sum of the square errors measured in $(y-direction)$ is minimum.

We shall study the best straight line through the given points.


f (x)
y=a+bx
X

$$Suppose\,that\ the\ best\ line\ is:$$

$$y = a + bx$$

$$e_i = (error\ at\ x = x_i) = y_i - (a + bx_i) = y_i - y$$

$$\sum e_i = \sum [y_i - (a + bx_i)]$$

$$\sum_{i=1}^{n} (e_i)^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

$$\sum_{i=1}^{n} (e_i)^2 = S$$

$$\therefore S = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

$$S=\sum_{i=1}^{n}[y_i-(a+bx_i)]^2$$

$$\frac{\partial S}{\partial a}=2[\sum_{i=1}^{n}\{y_i-(a+bx_i)\}](-1)=0$$

$$\sum_{i=1}^{n}[y_i-(a+bx_i)]=0$$

$$\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}(a+bx_i)=0$$

$$\therefore\sum_{i=1}^{n}y_i=\sum_{i=1}^{n}(a+bx_i)$$

$$\sum_{i=1}^{n}y_i=\sum_{i=1}^{n}a+\sum_{i=1}^{n}bx_i=a\sum_{i=1}^{n}1+b\sum_{i=1}^{n}x_i$$

$$\therefore\sum_{i=1}^{n}y_i=an+b\sum_{i=1}^{n}x_i\ldots\ldots\ldots(1)$$

$$S = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

$$\frac{\partial S}{\partial b} = 2[\sum_{i=1}^{n} \{y_i - (a + bx_i)\}](-x_i) = 0$$

$$\sum_{i=1}^{n} [y_i - (a + bx_i)]x_i = 0$$

$$\sum_{i=1}^{n} y_i\, x_i - \sum_{i=1}^{n} (a + bx_i)\, x_i = 0$$

$$\therefore \sum_{i=1}^{n} y_i\, x_i = \sum_{i=1}^{n} (a + bx_i)\, x_i$$

$$\sum_{i=1}^{n} y_i\, x_i = \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} bx_i^2$$

$$\therefore \sum_{i=1}^{n} y_i\, x_i = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 \ldots\ldots\ldots\ldots (2)$$

$$\therefore \sum_{i=1}^{n} y_i = an + b\sum_{i=1}^{n} x_i \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore \sum_{i=1}^{n} y_i\, x_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \ldots\ldots\ldots\ldots (2)$$

The Eqs. (1) and (2) can be written in matrix form:

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i x_i \end{bmatrix}$$

Example: Required *the best straight line passing the points*:

x_i :	0	1	3	4
y_i :	1	3	7	10

Solution:

	x_i	y_i	$y_i x_i$	x_i^2
	0	1	0	0
	1	3	3	1
	3	7	21	9
	4	10	40	16
$\sum$	8	21	64	26

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 21 \\ 64 \end{bmatrix}$$

$$4a + 8b = 21$$

$$8a + 26b = 64$$

$$Solve\ the\ \ above\ \ Eqs.\ getting\ \ a = 0.85\ \&\ b = 2.2$$

$$y = a + bx$$

$$\therefore\ y = 0.85 + 2.2x$$

Example: Fit the data shown below with a best line passing through the origin: $[y = bx]$.

x_i :	1	2	3	5
y_i :	2	5	7	9

$Solution:$

	x_i	y_i	$y_i x_i$	x_i^2
	1	2	2	1
	2	5	10	4
	3	7	21	9
	5	9	45	25
$\sum$	11	23	78	39

$$S = \sum_{i=1}^{n} [y_i - bx_i]^2$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} [y_i - bx_i]\,(-2x_i)$$

$$\sum_{i=1}^{n} y_i x_i = b \sum_{i=1}^{n} x_i^2$$

$$\therefore b = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2} = \frac{78}{39} = 2$$

$$\therefore y = 2x$$

$Example$: $Find\ the\ best\ curve\ (y = ae^{bx})\ passing$

$through\ the\ points$:

x_i:	0	1	3	6
y_i:	2	3	9	40

Solution: By taking Log. of both sides:

$$\ln y = \ln ae^{bx} = \ln a + \ln e^{bx}$$

$$Y = A + bx \quad \text{such that } (Y = \ln y \ \ \& \ \ A = \ln a)$$

	x_i	y_i	$Y_i = \ln y_i$	$x_i Y_i$	x_i^2
	0	2	0.693	0	0
	1	3	1.0986	1.0986	1
	3	9	2.197	6.591	9
	6	40	3.689	22.134	36
$\sum$	10		7.677	29.8236	46

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} * \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 46 \end{bmatrix} * \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7.677 \\ 29.8236 \end{bmatrix}$$

$$Solve\ the\ above\ Eqs.\ getting\ \ A = 0.6536 = \ln a \Rightarrow a = 1.9225$$

$$\& \ b = 0.5064$$

$$\therefore y = 1.9225 e^{0.5064x}$$

In the case of Quadratic Parabola:

$$y = a_o + a_1 x + a_2 x^2$$

$$e_i = y_i - (a_o + a_1 x_i + a_2 x_i^2)$$

$$\sum_{i=1}^{n} (e_i)^2 = S = \sum_{i=1}^{n} [y_i - (a_o + a_1 x_i + a_2 x_i^2)]^2$$

$$\frac{\partial S}{\partial a_o} = 0 \Rightarrow \sum_{i=1}^{n} y_i = n a_o + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 \ldots\ldots\ldots.(1)$$

$$\frac{\partial S}{\partial a_1} = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i = a_o \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 \ldots.(2)$$

$$\frac{\partial S}{\partial a_2} = 0 \Rightarrow \sum_{i=1}^{n} x_i^2 y_i = a_o \sum_{i=1}^{n} x_i^2 + a_1 \sum_{i=1}^{n} x_i^3 + a_2 \sum_{i=1}^{n} x_i^4 \ldots.(3)$$

In general:

$$y = a_o + a_1 x + a_2 x^2 + \dots\dots\dots + a_n x^n$$

$$\frac{\partial S}{\partial a_o} = 0 \qquad \dots\dots\dots\dots(1)$$

$$\frac{\partial S}{\partial a_1} = 0 \qquad \dots\dots\dots\dots(2)$$

$$\frac{\partial S}{\partial a_2} = 0 \qquad \dots\dots\dots\dots(3)$$

$$\vdots$$

$$\vdots$$

$$\frac{\partial S}{\partial a_n} = 0 \qquad \dots\dots\dots\dots(n+1)$$